Multipoint formulas in inverse scattering

Roman Novikov
École polytechnique, IETP RAS

Vladimir Sivkin
Lomonosov MSU

December 7, 2023

Overview

We present the first numerical study of multipoint formulas for finding leading coefficients in asymptotic expansions arising in potential and scattering theories. In particular, we implement different formulas for finding the Fourier transform of potential from the scattering amplitude at several high energies. We show that the aforementioned approach can be used for essential numerical improvements of classical results including the slowly convergent Born-Faddeev formula for inverse scattering at high energies. The approach of multipoint formulas can be also used for recovering the X-ray transform of potential from boundary values of the scattering wave functions at several high energies. Determination of total charge (electric or gravitational) from several exterior measurements is also considered. In addition, we show that the aforementioned multipoint formulas admit an efficient regularization for the case of random noise. This talk is based, in particular, on the work

R.G. Novikov, V.N. Sivkin, G.V. Sabinin, *Multipoint formulas in inverse problems and their numerical implementation,* Inverse Problems 39(12), 125016 (2023)

Asymptotic expansions

Many functions of scattering theory admit asymptotic expansions $z(s)$, $s \in (r, +\infty)$ of the form with leading coefficient

$$z(s) = a_1 + \frac{a_2}{s} + \ldots = \sum_{j=1}^{N} \frac{a_j}{s^{j-1}} + \mathcal{O}(s^{-N}), \tag{1}$$

see Atkinson 1949, Buslaev 1967, Melrose 1995, Yafaev 2003.

In addition, in some cases, the most important information is contained in a_1 (and/or some next leading terms). In this talk we present new results on finding a_1 from $z(s)$, given at several sufficiently large s, with applications to inverse scattering at high energies.

Examples

- Electrical or gravitational field with potential

$$U(x) = \int_D \frac{\rho(x')dx'}{|x - x'|}, \quad x \in \mathbb{R}^3, \tag{2}$$

where D is a bounded domain in $\mathbb{R}^3$. Then $sU(s\theta)$ admits multipole expansion of the form (1) with total charge leading coefficient

$$a_1 = \int_D \rho(x)dx. \tag{3}$$

We also consider the stationary Schrödinger equation

$$-\Delta\psi + v\psi = E\psi, \quad x \in \mathbb{R}^d, \; d \geq 2, \; E > 0,$$

Solutions describing scattering of planar wave

$$\psi^+(x,k) = \underbrace{e^{ikx}}_{\textit{planar wave}} + \underbrace{\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} c(d,|k|) \overbrace{f[v](k,|k|\frac{x}{|x|})}^{\textit{scattering amplitude}}}_{\textit{leading spherical wave}} + \mathcal{O}\left(\frac{1}{|x|^{(d+1)/2}}\right),$$

$|x| \to +\infty$ uniformly in $x/|x|$. The main scattering functions are:

- f - scattering amplitude (or far-field),
- ψ^+ - near-field,
- $|f|^2$ - differential scattering cross section,
- $|\psi^+|^2$ - amplitude of near-field.

Examples with far-field

- (Born, Faddeev, Buslaev, Melrose, Yafaev) For $v \in C_c^\infty(\mathbb{R}^d)$, scattering amplitude f has the expansion

$$f(s) = \widehat{v} + \sum_{j=2}^{N} \frac{a_j}{s^{j-1}} + \mathcal{O}(s^{-N}), \quad \text{as } s \to +\infty, \tag{4}$$

  where $s = \sqrt{E}$ is the energy level.

  Therefore, in this case the purpose is to reconstruct $\widehat{v}$ with good precision from scattering amplitude at **several** energy levels.

- Moreover, (4) has its phaseless analogue:

$$|f(s)|^2 = |\widehat{v}|^2 + \sum_{j=2}^{N} \frac{a_j}{s^{j-1}} + \mathcal{O}(s^{-N}), \quad \text{as } s \to +\infty. \tag{5}$$

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Here

$$f(s) = f(k_s, l_s), \quad \hat{v} = \hat{v}(p), \quad a_j = a_j(p, \omega),$$
$$k_s = p/2 + (E - p^2/4)^{1/2}\omega, \quad l_s = -p/2 + (E - p^2/4)^{1/2}\omega,$$
$$E = E(s) = s^2, \quad p \in \mathbb{R}^d, \quad p \cdot \omega = 0, \quad \omega \in \mathbb{S}^{d-1}.$$

Example with near- to far-field

- Consider the scattering solutions ψ^+ at fixed enery. Then, according to Atkinson-Wilson type expansion:

$$\psi^+(x,k) = e^{ikx} + \frac{e^{i|k|s}}{s^{(d-1)/2}} \left(f_1 + \sum_{j=2}^{N} \frac{f_j}{s^{j-1}} + \mathcal{O}\left(\frac{1}{s^N}\right)\right), \quad s \to +\infty, \qquad (6)$$

where $s = |x|$.

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where $s = |x|$. Here

$$f_j = f_j(k, |k|\hat{x}), \quad x = s\hat{x}, \quad f_1(k,l) = c(d,|k|) f(k,l),$$
$$c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}, \text{ for } \sqrt{-2\pi i} = \sqrt{2\pi} e^{-i\pi/4},$$

where $\hat{x}$, k, $l \in \mathbb{R}^d$, and $|k|^2 = |l|^2 = E$, $|\hat{x}|^2 = 1$.

Examples with near-field

- (Yafaev, Klibanov, Romanov) Consider the scattering solutions ψ^+ of Schrödinger equation at fixed point. For $v \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have that

$$\psi^+(x,k) = e^{ikx}\left(1 + \sum_{j=1}^{N-1} \frac{b_j(x,\theta)}{s^j} + \mathcal{O}(s^{-N})\right), \text{ as } s \to +\infty, \tag{7}$$

$$b_1(x,\theta) = \frac{1}{2i} Dv(x,-\theta), \quad Dv(x,\theta) := \int_0^{+\infty} v(x+\tau\theta)d\tau, \tag{8}$$

where $x \in \mathbb{R}^d$, $s = |k| = \sqrt{E}$ is energy level, $\theta = k/|k|$ (x and θ are fixed). Note that Dv is known as the divergent beam transform of v, and is very popular in tomography.

Reconstruction

So, we consider

$$z(s) = a_1 + \frac{a_2}{s} + \dots = \sum_{j=1}^{N} \frac{a_j}{s^{j-1}} + \mathcal{O}(s^{-N}). \tag{9}$$

Let $z = z(s)$ be given at n points $s + \tau_1 < s + \tau_2 < \dots < s + \tau_n$.

The simplest reconstruction of a_1, which is widely used in direct and inverse scattering, gives

$$a_1 = z(s + \tau_n) + \mathcal{O}(s^{-1}), \quad s \to +\infty.$$

However, the error of reconstruction has a slow decay as $\mathcal{O}(s^{-1})$.

Reconstruction

In turn, according to (Novikov) a_1 can be found with high accuracy from z given at $s+\tau_1 < s+\tau_2 < \dots < s+\tau_n$ via formulas

$$\begin{aligned} a_1 &= \sum_{j=1}^{n} y_j(s,\vec{\tau}) z(s+\tau_j) + \mathcal{O}(s^{-n}), \ \ s \to +\infty, \\ y_j(s,\vec{\tau}) &= (-1)^n \frac{(s+\tau_j)^{n-1}}{\alpha_j(\vec{\tau})\beta_{n,j}(\vec{\tau})} \sim s^{n-1}, \\ \alpha_j(\vec{\tau}) &= \Pi_{i=1}^{j-1}(\tau_j-\tau_i), \quad \beta_{n,j}(\vec{\tau}) = \Pi_{i=j+1}^{n}(\tau_i-\tau_j). \end{aligned} \tag{10}$$

However, we found that these formulas are unstable to noise.

Unstability

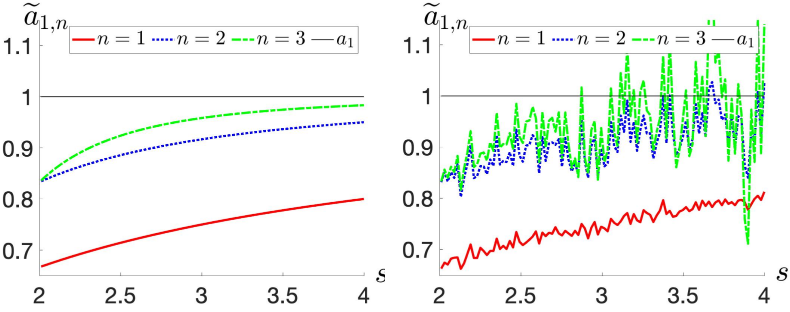
Indeed, if data are given with random noise:

$$z^{noisy}(s) = z(s) + \varepsilon\mathcal{N}(0,1)$$

with dispersion ε^2, then the dispersion of reconstruction grows rapidly for high s even for small $n \geq 2$:

$$\mathbb{D}(a_{1,n}) \sim \varepsilon^2 s^{2(n-1)}.$$

Unstability



(a) Exact data. (b) Noisy data.

Figure: [NSS2023] n-point reconstructions $a_{1,n}(s)$ of $a_1 = 1$ for $z(s) = s/(s+1)$ with $\tau_j = j - 1$. (a) Two- and three-point formulas rapidly converge to exact value. (b) Two- and three-point formulas are unstable to noise.

Regularization

In order to make multipoint formulas appropriate for applications, we propose a regularisation method with a parameter r. We construct

$$\tilde{a}^r_{1,n} = \sum_{j=1}^{n} y^r_j(s, \vec{\tau}) z(s_j(s)), \tag{11}$$

where $y^r = (y^r_1, ..., y^r_n)$ depends only on n and r, and $r \in [n^{-1/2}, \|(y_{1,n}, ..., y_{n,n})\|]$. In particular, $\|y^r\| = r$.

Note that, the reconstruction $\tilde{a}^r_{1,n}$ has the following properties:

- the dispersion of reconstruction $a^r_{1,n}$ from noisy data is bounded by

$$\mathbb{D}(a^r_{1,n}(s, \vec{\tau})) \leq r^2 \varepsilon^2 \quad \text{independently of } s; \tag{12}$$

- for the noiseless function z, the regularized reconstruction is the best possible under the condition $\|y^r\| \leq r$.

Regularization

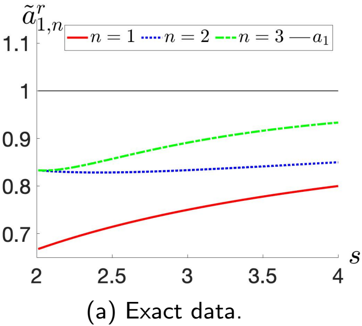
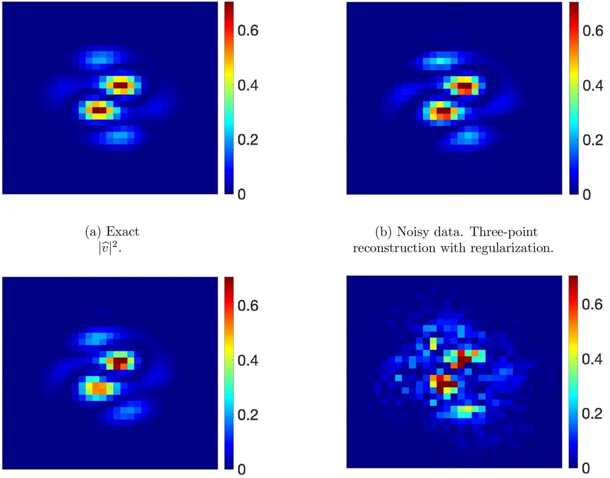


Figure: [NSS2023] n-point regularized reconstructions $a^r_{1,n}(s)$ of $a_1 = 1$ for $z(s) = s/(s+1)$ with $\tau_j = j - 1$. Regularization parameter $r = \sqrt{5}$. (a) Two- and three-point regularized formulas converge to exact value, but not so rapid as in Figure 1(a). (b) Two- and three-point regularized formulas are stable to noise.



(a) Exact $|\widehat{v}|^2$.

(b) Noisy data. Three-point reconstruction with regularization.

(c) Exact data. One-point reconstruction.

(d) Noisy data. Three-point reconstruction without regularization.

Figure: Exact $|\widehat{v}|^2$ and examples of its phaseless inverse scattering reconstructions from phaseless scattering amplitude $|f|^2$.

Conclusions

This work contributes to studies on phase retrieval, phaseless inverse scattering, and multipoint formulas in inverse problems, and includes the following result: We give the first numerical implementation of the method of multipoint formulas in inverse problems, including an efficient regularization of these formulas for the noisy case. By this method we also obtain new theoretical results on polychromatic inverse scattering problems (from far-field data, phaseless far-field data, and from near-field data).

Thank you for attention!