Reconstruction from truncated Fourier transform

Roman Novikov

joint works with Mikhail Isaev and Grigory Sabinin

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We give formulas for finding a compactly supported function v on $\mathbb{R}^d$, $d \geq 1$, from its Fourier transform $\mathcal{F}v$ given within the ball B_r. For the one-dimensional case, these formulas are based on the theory of prolate spheroidal wave functions. In multidimensions, well-known results of the Radon transform theory reduce the problem to the one-dimensional case.

We also present a numerical implementation of these results. In particular, the results obtained give super-resolution reconstruction, that is, they allow recovering details beyond the diffraction limit, that is, details of size less than π/r, where r is the radius of the ball mentioned above.

This talk is based on the works:

[IN] M. Isaev, R.G. Novikov, Reconstruction from the Fourier transform on the ball via prolate spheroidal wave functions, Journal de Mathématiques Pures et Appliquées 163 (July), 318-333 (2022)

[INS] M. Isaev, R.G. Novikov, G.V. Sabinin, Numerical reconstruction from the Fourier transform on the ball using prolate spheroidal wave functions, Inverse Problems 38(10), 105002 (2022)

1. Basic problems

We consider the Fourier transform F defined by the formula

$$\mathcal{F}[v](p) = \hat{v}(p) := \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{ipq} v(q) dq, \qquad p \in \mathbb{R}^d, \qquad (1)$$

where v is a complex-valued test function on $\mathbb{R}^d$, $d \geq 1$.
Let $B_\rho := \{q \in \mathbb{R}^d : |q| < \rho\}$, $\rho > 0$.

Problem 1. Find $v \in L^2(\mathbb{R}^d)$, where $supp\, v \subset B_\sigma$, from $\hat{v} = \mathcal{F}v$ given on the ball B_r (possibly with some noise), for fixed $r, \sigma > 0$.

Problem 1 arises in different areas such as Fourier analysis, linearized inverse scattering and image processing, and has been extensively studied in the literature. Solving Problem 1 is complicated considerably by the fact it is exponentially unstable, for fixed $r, \sigma > 0$. Nevertheless, there exist several techniques to approach this problem theoretically and numerically; see, [IN], [INS] and references therein.

The conventional approach for solving Problem 1 is based on the following approximation

$$v \approx v_{\mathrm{naive}} := \mathcal{F}^{-1}[w](q) = \int\limits_{B_r} e^{-ipq} w(p) dp \qquad q \in B_{\sigma}, \qquad (2)$$

where $\mathcal{F}^{-1}$ is the standard inverse Fourier transform and w is such that $w|_{B_r}$ coincides with the data of Problem 1 and $w|_{\mathbb{R}^d \setminus B_r} \equiv 0$.

Formula (2) leads to a stable and accurate reconstruction for sufficiently large r. However, it has well-known diffraction limit: small details (especially less than π/r) are blurred. A new approach for *super-resolution* in comparison with the resolution of (2) was recently developed in [IN], [INS].

2. Preliminaries

For convenience, we consider the scaling of v with respect to the size of its support:

$$v_\sigma(q) := v(\sigma q), \qquad q \in \mathbb{R}^d. \tag{3}$$

Note that $\mathit{supp}\, v_\sigma \subset B_1$.
Let

$$c := r\sigma \qquad (4).$$

The data in Problem 1 (for the case without noise) can be presented as follows (see [IN]):

$$\hat{v}(rx) = \frac{\sigma}{2\pi}\mathcal{F}_c\left[v_\sigma\right](x) \qquad \text{for } d = 1, \tag{5}$$

$$\hat{v}(rx\theta) = \left(\frac{\sigma}{2\pi}\right)^d \mathcal{F}_c\left[\mathcal{R}_\theta[v_\sigma]\right](x) \qquad \text{for } d \geq 2, \tag{6}$$

where $x \in [-1, 1]$, $\theta \in \mathbb{S}^{d-1}$, $c = r\sigma$, $v_\sigma(q) = v(\sigma q)$, the operators $\mathcal{F}_c$ and $\mathcal{R}_\theta$ are defined by

$$\mathcal{F}_c[f](x) := \int_{-1}^{1} e^{icxy} f(y)dy, \qquad x \in [-1, 1], \tag{7}$$

$$\mathcal{R}_\theta[u](y) := \int\limits_{q\in\mathbb{R}^d, q\theta = y} u(q)dq, \qquad y \in \mathbb{R}, \tag{8}$$

where f is a test function on $[-1, 1]$ and u is a test function of $\mathbb{R}^d$.

Recall that $\mathcal{R}_\theta[u] \equiv \mathcal{R}[u](\cdot, \theta)$, where $\mathcal{R}_\theta$ is defined by (8) and $\mathcal{R}$ is the classical Radon transform. In fact, presentation (6) follows from the projection theorem of the Radon transform theory.

The operator $\mathcal{F}_c$ defined by (7) is a variant of band-limited Fourier transform. This operator is one of the key objects of the theory of *prolate spheroidal wave functions*. In particular, the operator $\mathcal{F}_c$ has the following singular value decomposition in $L^2([-1, 1])$:

$$\mathcal{F}_c[f](x) = \sum_{j \in \mathbb{N}} \mu_{j,c} \psi_{j,c}(x) \int_{-1}^{1} \psi_{j,c}(y) f(y) dy, \tag{9}$$

where $(\psi_{j,c})_{j \in \mathbb{N}}$ are the prolate spheroidal wave functions and the eigenvalues $\{\mu_{j,c}\}_{j \in \mathbb{N}}$ satisfy

$$0 < |\mu_{j+1,c}| < |\mu_{j,c}| \text{ for all } j \in \mathbb{N} = \{0, 1, 2 \ldots\}, \tag{10}$$

$$\left\lfloor \frac{2c}{\pi} \right\rfloor - 1 \leq \left| \{ j \in \mathbb{N},\ |\mu_{j,c}| \geq \sqrt{\pi/c} \} \right| \leq \left\lceil \frac{2c}{\pi} \right\rceil + 1, \qquad (11)$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and the ceiling functions, respectively, and $|\cdot|$ is the number of elements in a set,

$$\mu_{j,c} \text{ decay superexponentially as } j \to \infty. \qquad (12)$$

The functions $(\psi_{j,c})_{j\in\mathbb{N}}$ are certain of wave functions introduced by Niven in 1880 for solving the Helmholtz equation in prolate spheroidal coordinates. Originally, $(\psi_{j,c})_{j\in\mathbb{N}}$ are defined as the eigenfunctions of the spectral problem

$$\frac{d}{dx}\left[(1-x^2)\frac{d\psi}{dx}\right]+c^2x^2\psi=\chi\psi, \qquad \psi\in C^2([-1,1]).$$

The fact that $(\psi_{j,c})_{j\in\mathbb{N}}$ are the eigenfunctions of the finite Fourier transform $\mathcal{F}_c$ defined by (7) was pointed out by Slepian and Pollak in 1961 as a special case of more general integral relations satisfied by Niven's wave functions. As mentioned in [Slepian, Pollak, 1961]

> *"These functions ... possess properties that make them ideally suited for the study of certain questions regarding the relationship between functions and their Fourier transforms."*

3. Reconstruction formulas from [IN]

For $d = 1$:

$$v_\sigma = \frac{2\pi}{\sigma}\mathcal{F}_c^{-1}[\hat{v}_r], \tag{13}$$

where $\hat{v}_r(x) = \hat{v}(rx)$, $x \in [-1, 1]$,

$$\mathcal{F}_c^{-1}[g](y) = \sum_{j\in\mathbb{N}} \frac{1}{\mu_{j,c}}\psi_{j,c}(y)\int_{-1}^{1}\psi_{j,c}(x)g(x)dx, \tag{14}$$

g is a test function from the range of $\mathcal{F}_c$ acting on $L^2([-1, 1])$.
For $d \geq 2$:

$$v_\sigma = \left(\frac{2\pi}{\sigma}\right)^d \mathcal{R}^{-1}[f_{r,\sigma}], \tag{15}$$

$$f_{r,\sigma}(y, \theta) = \mathcal{F}_c^{-1}[\hat{v}_{r,\theta}](y), \text{ if } y \in [-1, 1], \text{ and } f_{r,\sigma} = 0 \text{ otherwise },$$

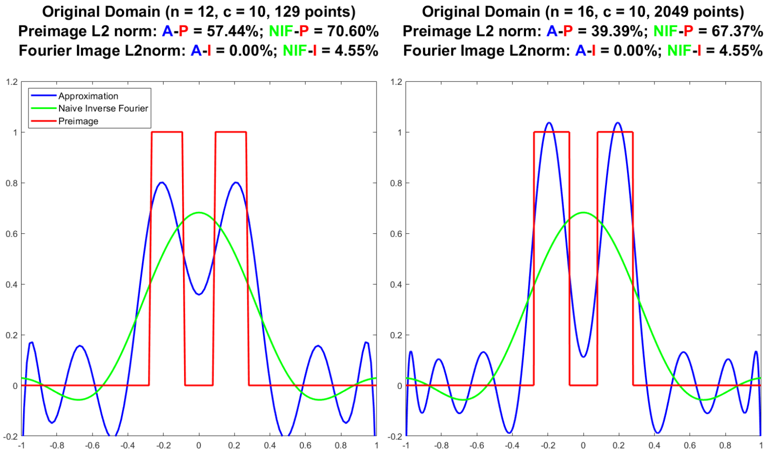
$$\hat{v}_{r,\theta}(x) = \hat{v}(rx\theta), \qquad x \in [-1, 1],\ \theta \in \mathbb{S}^{d-1}.$$

For the case of noisy data in Problem 1, the operator $\mathcal{F}_c^{-1}$ is approximated by the finite rank operator $\mathcal{F}_{n,c}^{-1}$ defined by

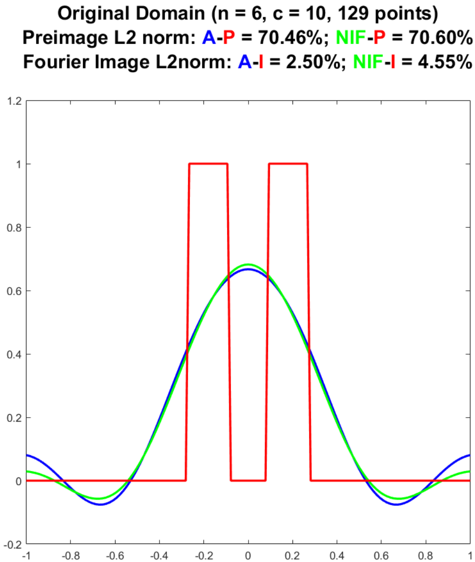
$$\mathcal{F}_{n,c}^{-1}[g](y) := \sum_{j=0}^{n} \frac{1}{\mu_{j,c}} \psi_{j,c}(y) \int_{-1}^{1} \psi_{j,c}(x) g(x) dx. \qquad (16)$$

The operator $\mathcal{F}_{n,c}^{-1}$ is correctly defined on $L^2([-1,1])$ for any $n \in \mathbb{N}$. In addition, $\mathcal{F}_{n,c}^{-1}[g]$ is *the quasi-solution* in the sense of Ivanov of the equation $\mathcal{F}_c[f] = g \in L^2([-1,1])$ on the span of the first $n+1$ functions $(\psi_{j,c})_{j \leq n}$.
The rank n is a regularisation parameter.

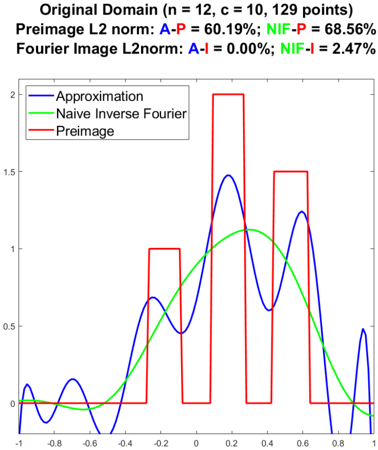
Numerical results / 1D case / Two stairs

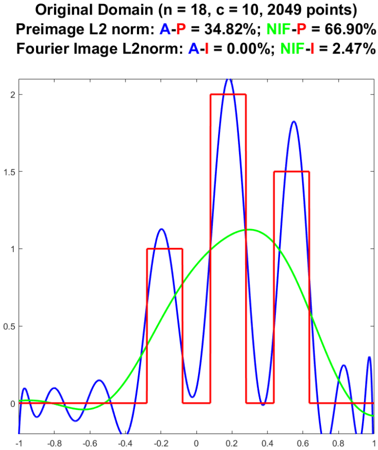


$n = n_0 := \left\lfloor \frac{2c}{\pi} \right\rfloor$: similarity with NIF

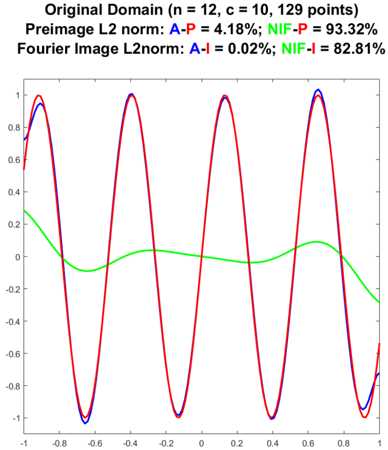


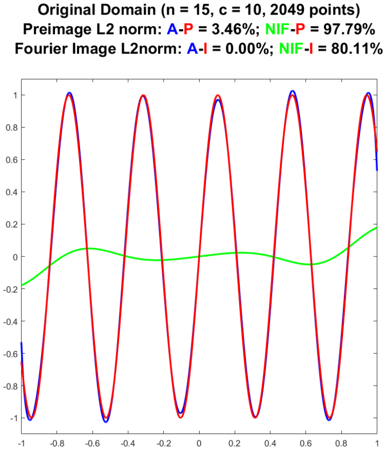
Numerical results / 1D case / Three stairs



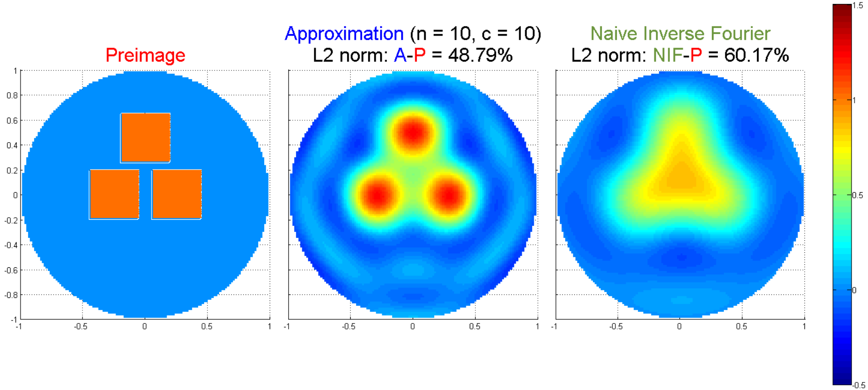


Numerical results / 1D case / Sin(12x) and sin(15x)



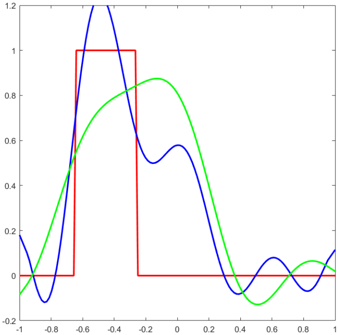


Numerical results / 2D case / Three stairs

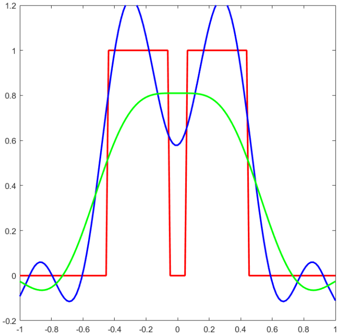


Numerical results / 2D case / Three stairs (cross-sections)

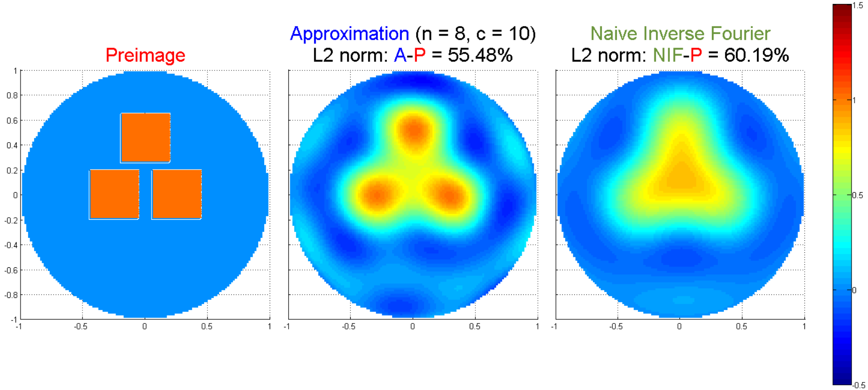
Original Domain (n = 10, c = 10), Vertical CS (90 degrees)
CS L2 norm: A-P = 62.89%, NIF-P = 92.29%



Original Domain (n = 10, c = 10), Vertical CS (0 degrees)
CS L2 norm: A-P = 39.55%, NIF-P = 48.81%



Numerical results / 2D case / Three stairs (21% of $\mathcal{L}^2$ noise)



Summary

- In spite of the *exponential instability* of the problem, we achieved *super-resolution* even for noisy data by appropriate choice of the regularisation parameter n. In particular, for $d \geq 2$, the approach works well even for a *considerable level of random noise*.
- Our reconstruction (with appropriate choice of n) gives *smaller errors* in $\mathcal{L}^2$-norm (in both Fourier domain and spatial domain) than the *conventional* reconstruction.
- Our reconstruction with $n = n_0 := \left\lfloor \frac{2c}{\pi} \right\rfloor$ behaves *similarly* to the *conventional* reconstruction. In our examples, taking n larger than n_0 gives better results.

We expect that similar numerical behaviour (in particular, *super-resolution*) is also possible for *monochromatic inverse scattering* and for other generalisations of the considered problem.