УДК 550.34

RAYLEIGH WAVES WITHOUT CUBIC EQUATIONS

L. Knopoff

Department of Physics and Astronomy and Institute of Geophysics and Planetary Physics
University of California, Los Angeles, USA

One of the major contributions of Volodya Keilis-Borok has been the founding and overseeing of the publication of *Computational Seismology.* This journal has been an important resource for solutions to difficult mathematical and computational problems in seismicity and seismic wave propagation.

ВОЛНЫ РЭЛЕЯ БЕЗ КУБИЧЕСКИХ УРАВНЕНИЙ

Л. Кнопов

Калифорнийский университет в Лос-Анджелесе, Департамент физики и астрономии,
Институт физики Земли и планет, Лос-Анджелес, США

Один из главных вкладов Володи Кейлис-Борока – это основание ежегодного сборника *Вычислительная сейсмология* и руководство его изданием. Этот сборник послужил важным ресурсом в решении трудных математических и вычислительных задач сейсмичности и распространения сейсмических волн

The physics of seismology forms the basis for any mathematical formulation of a computational problem. There are two motivations for the introduction of deeper physics than the most superficial statements into computational seismology. In one case, the physics that is used in the mathematical constructions may be so simple, that it leads to models that may have fit only part of the complete suite of geophysical observations. In this case one might wish to use more sophisticated physics to constrict an unmanageable manifold of models in the hope that extrapolations from appropriate models can be better used in a predictive sense. As an example we cite recent interest in the understanding of the processes that lead to the Gutenberg-Richter (GR) frequency-moment or frequency-energy scale-independent relation for regional earthquake occurrence. Any scale-independent model will yield scale-independent distributions such as the GR law. Of course one can restrict the models by the use of data with greater numbers of degrees of freedom, i.e. to search for deviations from scale-independent behavior. One can also restrict the class of scale-independent models to give the appropriate b-value, or to account for broad-band seismographic observations. On the physics side, one might improve models of fracture by introducing dynamics into these simulations, or by introducing inhomogeneities into the description of faulting. A second motivation for considering physics in the computational process, is to provide understanding to discoveries that are cloaked in either phenomenology or mathematics. I can summarize this class of problems by suggesting that there is an innate curiosity on the part of most scientists to ask why or how certain observations in mathematical theory or in correlation studies arise. As an example of the phenomenological or correlation genre, we might ask why large earthquakes cluster in space and in time. The issues of understanding of physical underpinnings are self-indulgent exercises without immediate consequences on the practicalities of computation. A focus on the “why or how” problem is an issue of pedagogy rather than one of practicality. In this paper I give an example of the search for a physical basis for a mathematical problem in seismology. I seek to explore the logical space between the formulation and the numerical results of a problem in the study of Rayleigh waves.

The beginning student of seismology often asks about the physical nature of Rayleigh waves on a half-space. The professorial response to the physics question is usually a mathematical one, which is of course the answer to a different question: for example, it is replied that the Rayleigh wave on a homogeneous half-space is the solution to an eigenvalue equation, with an eigenvalue that is one root of a cubic equation; or there is a response in terms of the reflection at complex angles of plane P- and S-waves that are incident on the surface. The student questions why a direct physical explanation in terms of the deformation of an elastic body is not possible, in direct contrast to the immediacy and transparency with which compression waves and shear waves can be understood in terms of a transparent physical picture of the deformation of elastic materials under stress. Why must the nature of a Rayleigh wave be obscured in abstract mathematical manipulations? Is there no simple explanation in terms of the physics of such a fundamental wave property, in analogy with that for compression and shear?

In this note, I offer a simple physical explanation for the phenomenon of Rayleigh waves on a homogeneous half-space. I show without difficult mathematics that the motions will be retrograde elliptical on the surface for sinusoidal time dependence of the Rayleigh wave motions and prograde elliptical at depth. Since this will be a qualitative theory, it will be impossible to derive the phase velocity of Rayleigh waves precisely, but I show that the phase velocity is less than the S-wave velocity.

Consider an elastic half-space deformed by horizontal displacements at the surface as illustrated in Fig. 1. To the left of point A, the surface is displaced uniformly to the right, while to the right of point B, the surface is displaced equally uniformly to the left. Since there is no compression or extension of the undeformed regions at the far left and right, there is of course no vertical component of the displacement in these regions. A vertical displacement of the surface in the central region is associated with the gradient of the horizontal component of the displacement.

We solve for the sign of the vertical component of the displacement in the transitional interval between the two undeformed regions. Consider the point X at the surface of the half-space (Fig. 1).

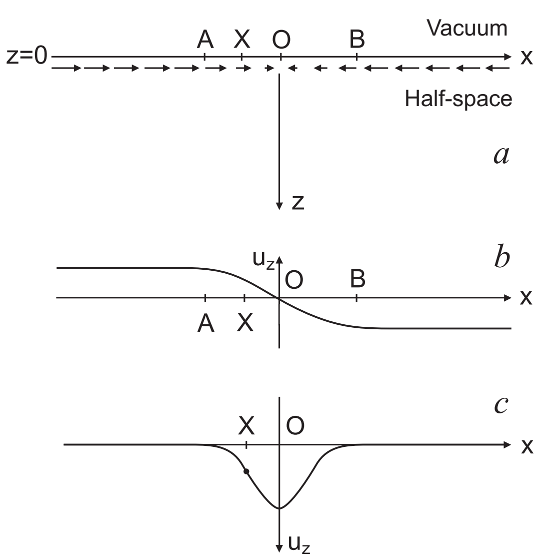

Fig. 1. Elastic half-space deformed by horizontal displacements at the surface.

a) Horizontal displacements applied at the surface of an elastic half-space. The displacements are positive and uniform to the left of A and negative and uniform to the right of B; *b*) $u_x(x)$ at $z = 0$. *c*) $u_z(x)$ at $z = 0$

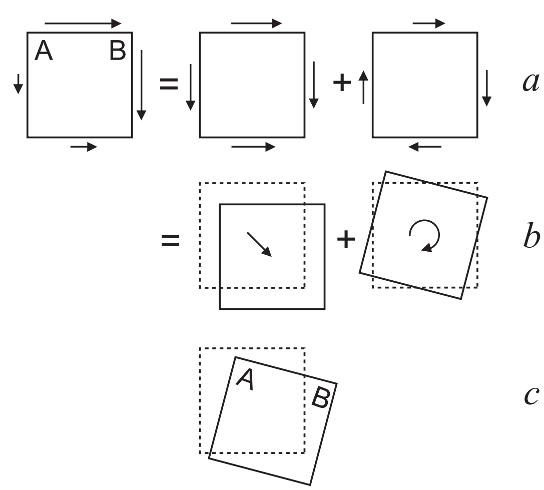

Fig. 2. Deformation of an element of area at point X (see Fig. 1) in the surface of the half-space, under a condition of vanishing shear stress.

a) Deformation has a negative vertical gradient of u_x and an equal positive horizontal gradient of u_z. These can be decomposed into a translation and a rotation; *b*) vectorial decomposition as in *a*); *c*) total motion of element showing depression and rotation of surface line element AB

The displacements of the elastic medium resulting from the deformation of the surface must decrease with increasing depth, and vanish at infinite depth. Thus $u_{x,z} < 0$. (I use the comma notation for partial derivatives, $u_{x,z} = \partial u_x/\partial u_z$, etc.) Since the boundary condition at the surface is $\tau_{xz}/\mu = u_{x,z} + u_{z,x} = 0$, then $u_{z,x} > 0$ directly. Thus site X must be depressed and hence the horizontal compression in the region between A and B must have a corresponding indentation of the surface. We can construct a geometrical view of the same observation (Fig. 2). Consider an element of area at the surface at point X of Fig. 1. As above, because of the negative gradient $\partial u_x/\partial u_z$, there must be an equal positive gradient $\partial u_z/\partial u_x$ under the condition of zero shear strain at the surface. These gradients are shown at the left in Fig. 2*a*. After we subtract the mean motion (Fig. 2*a*) which is rightward and downward (Fig. 2*b*), there remains a residual net rotation of the elemental square in the clockwise direction. Hence the surface is indented at X (Fig. 2*c*). The problem of the wrinkling of a carpet under horizontal displacements is somewhat different from the one considered here, since the carpet problem involves a lower boundary that is not fixed; the carpet cannot undergo a vertical depression.

Our intuitive expectation that the surface under compression would be likely to bulge upward would be correct if the half-space were an incompressible fluid. But in the present case, it is indented because the elastic half-space is not incompressible and because the shear stress vanishes at the surface.

We turn to the problem of the deformation of the half-space by a spatially sinusoidal set of horizontal displacements at the surface (Fig. 3,*a*). From the argument above, the vertical component of the displacement must also be sinusoidal; the maxima of the bulges B and the minima of the indentations D in the vertical component of the displacement at the surface must correspond to the extreme values of extension E and compression C in the gradient of the horizontal component of the displacement respectively (Fig. 3,*b*). A zero value of the vertical component of the displacement corresponds to a zero value of the gradient of the horizontal component. The two components of the sinusoids are 90° out of phase at the surface. Since we are describing a wave, the temporal phase relationship between the two components is also 90°. Because of the phase shift between the two components of the motion, the motion at the surface must be elliptical. From the bulge/extension and indentation/compression phase correlations at times (a,b,c,d), it is trivial to demonstrate that the elliptical motion is retrograde at the surface (Fig. 3,*c*); we say nothing quantitative about the ellipticity. Note that for waves that propagate to the right, the wave u_z leads the wave u_x, while for waves that propagate to the left, the wave u_x leads the wave u_z. The seeming anisotropy is an illusion of the vectorial nature of the motion: Bulges in the component u_z are always located at sites of extension in the component u_x, and similarly for indentations and compressions for either direction of propagation. The motions are

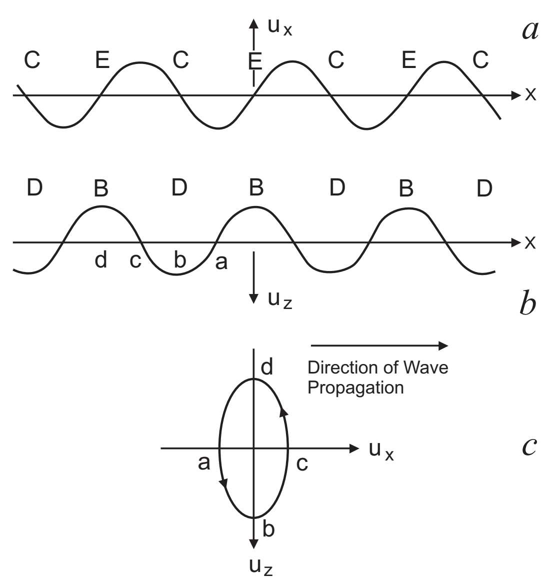

Fig. 3. Deformation of the half-space by sinusoidal displacements at the surface.
a) Sinusoidal deformation of the surface by horizontal components of motion. C and E are sites of maximum compression and extension; *b*) schematic vertical component of motion in correspondence with *a*. D and B denote sites of maximum indentation (depression) and bulge. These correspond identically to sites C and E in *a*; *c*) motion is retrograde elliptically polarized. Successive times a, b, c, d correspond to those in *b*) for a wave moving to the right

retrograde elliptical at the surface for either direction of propagation.

Up to this point, none of these arguments have focused on the issues of the wave nature of Rayleigh waves, except for the mention of the retrograde nature of the motion at the surface for sinusoidal excitation. To evaluate the remaining remarkable property of Rayleigh waves, namely that its wave velocity is less than that of S-waves, we must introduce the essential ingredient that characterizes the differences between statics and dynamics, namely inertia. The introduction of inertia gives rise to the wave properties of the deformation, characterized by the wave equation; stress is introduced into the discussion implicitly, insofar as the wave equation is derived from the gradient of the stress in the usual way. If displacements are applied at the free surface and not elsewhere, then the displacements must decrease with depth at sufficient depth and approach zero at infinite depth within the half-space. For sinusoidal motions, the displacements must decrease with depth exponentially at a rate that depends on the wavelength of the horizontally traveling waves; the rates of decrease are different for the shear and compressional components of the stress fields. Thus the solution to the wave equation must be a harmonic wave that travels in the x-direction and decays exponentially in the z-direction at the rate η when scaled by the frequency ω,

$$e^{-\omega\eta z}e^{i\omega(x/c-t)},$$

where c is the velocity of the Rayleigh wave; here the vertical spatial decay rate η is η_S or η_P depending on whether we consider the shear or compression wave components of the deformation. After substitution into the wave equation, we get

$$\frac{1}{c^2}=\frac{1}{(\alpha^2,\beta^2)}+\eta^2_{(P,S)}$$

where α and β are the P- and S-wave velocities. By inspection, $c<\beta<\alpha$. Thus the property that the phase velocity of Rayleigh waves is less than the S-wave velocity is a direct consequence of the vanishing of the motion at infinite depth.

It follows from the above that $\eta_P>\eta_S$, i.e. that the P-wave component of the motion decays more rapidly than the S-wave component. The contrast between the two decay rates becomes very large as $c\to\beta$. The elastic features of the Rayleigh wave, including displacement vectors and stress and strain components, all propagate with phase velocity c, and all have amplitudes that vary with depth as a sum of terms with the two depth decay rates,

$$\left(Ae^{-\omega\eta_S z}+Be^{-\omega\eta_P z}\right)e^{i\omega(x/c-t)}.$$

The exception to this uniform behavior is the volumetric strain $\varepsilon_{kk}=\varepsilon_{xx}+\varepsilon_{zz}$, which has only the single depth decay component $e^{-\omega\eta_P z}$, a result that is easily understood because the volumetric strain is the invariant trace of the tensor and hence its falloff rate cannot depend on the shear properties of the system, i.e. it is a solution to the P-wave equation only. There is a 90° phase shift between the components of the pairs (u_x,u_z), $(\varepsilon_{xx},\varepsilon_{xz})$, etc.

Because the volumetric strain falls off more rapidly with depth than the shear strain, the wave motions at sufficiently great depth are dominated by the shear properties. Of course, anywhere on the axis below a symmetric extremum of indentation at the surface there can be no shear strain. Since the material at great depth behaves like an incompressible fluid, it follows that the horizontal component of the motion near the axis of symmetry must diverge outward from the axis of compression, i.e. under an indentation at the surface, with the converse under a bulge (Fig. 4). Hence there is a reversal of the horizontal component of the motion at depth relative to that at the surface, and hence the elliptical polarization at depth must be prograde. As a consequence, there must be a crossover depth at which the horizontal component of sinusoidal motions is zero. This argument cannot be made near the surface, since there the motion is controlled by the vanishing of the shear strain at the surface; the volumetric strain does not vanish at the surface as it does at depth, in comparison with the shear strain.

We can now argue, again qualitatively, that the velocity of Rayleigh waves should be close to the S-wave velocity; we have already indicated that it must be less than the S-wave velocity. The velocity of the small motions in the Rayleigh waves at great depth is of course the same as that of the motions at the surface. But the motions at great depth are mainly shear motions, since the compressional part of the motion is relatively small at these depths. Thus the phase velocity of the motion must be close to the S-wave velocity, and less than it by the argument above. The small difference between the Rayleigh- and S-wave velocities is of course due to the small amount of energy remaining in the compressional strain at these depths. Realistically, it is the large deformation at the surface that drags the motion at depth along with it; after all, it is the boundary conditions at the surface that determine the velocity of Rayleigh waves, and a velocity determined from the (mainly) shear properties at depth is merely a method of qualitative calculation, and is not the principal physical cause of the sub-shear wave velocity.

There is a final point concerning the rate of decrease of the vertical component of the motions. The normal stress $\tau_{zz} = \lambda(u_{x,x} + u_{z,z}) + 2\mu u_{z,z} = 0$ at the surface. Hence

$$u_{z,z} = -\frac{\lambda}{\lambda + 2\mu} u_{x,x}$$

at $z = 0$. Since $u_{x,x} < 0$, then $u_{z,z} > 0$. Thus not only is $u_z > 0$, but also $\partial u_z / \partial u_z > 0$ below a region of horizontal compression at the surface. Thus the vertical displacement actually increases with depth immediately below the surface; of course it decays to zero at greater depth. The general shape of the depth dependence in both components of the motion can now be sketched as in Fig. 5.

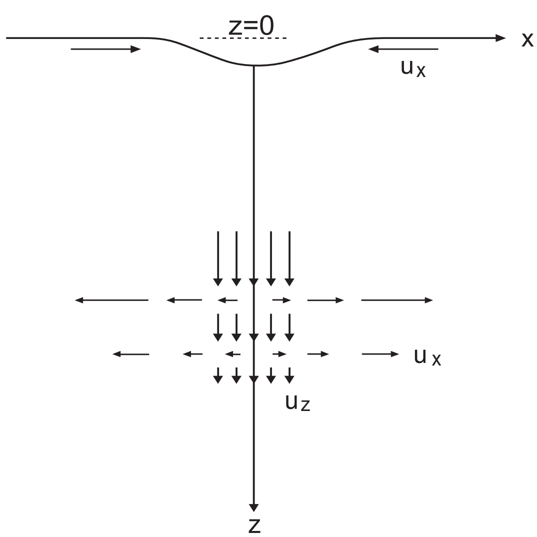

Fig. 4. Motion at depth near the axis of symmetry under a depression at the surface. Because of the vertical compression at depth, the horizontal components of motion diverge from the axis under the condition of incompressiblity at depth. Thus there is a reversal in sign of u_x at great enough depth

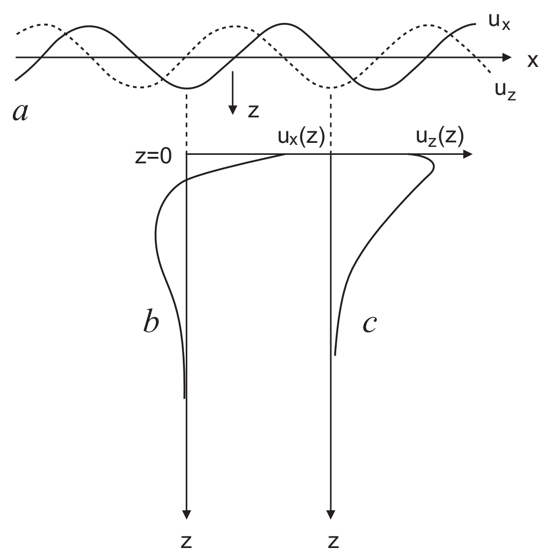

Fig. 5. Depth dependence of motion.
a) Schematic horizontal and vertical components of the motion at $z = 0$, as in Fig. 3; *b*) schematic amplitude $u_x(z)$ beneath a maximum of u_x, showing reversal of sign at depth; *c*) schematic amplitude $u_z(z)$ beneath a maximum of u_z, showing positive gradient near the surface

I have benefited significantly from the valuable, critical suggestions of Professor Chen Yun-tai.